Viscous Fingering: Theory, Simulation & Modelling

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Viscous fingering

\[ \frac{\partial c}{\partial n} = 0, \quad \frac{\partial p}{\partial n} = 0 \]

Contents

- The Peaceman Theory
- The moving point method
- A very simple theory
- Homogenisation
Peaceman’s miscible displacement equations

\[
\frac{\partial c}{\partial t} + \nabla \cdot (uc) = \nabla \cdot (D \nabla c), \quad 0 \leq c \leq 1
\]

\[\nabla \cdot u = 0\]

\[u = -\frac{k(x) \cdot \nabla p}{\mu(c)}\]

\[\mu(c) = \mu_o(1 - c) + \mu_s c, \quad \mu_s \ll \mu_o\]

The Moving Point Method

- Gardner, Peaceman and Pozzi, SPE, 1964 (GPP)

\[\nabla \cdot u = 0\]

\[u = -\frac{k(x) \cdot \nabla p}{\mu(c)}\]

\[
\frac{dx_p}{dt} = \frac{u(x_{p,t})}{\phi(x_p)}, \quad \frac{dC_p}{dt} = \nabla (D \nabla c)|_{x=x_p}, \quad c_i^n = \sum_{\text{points in the cell}} C_p
\]
**Diffusion & Convergence**

- GPP 64
  - Central Differences on cells

- CLF 85
  - Central Differences on moving points

- Rees & Morton 91
  - Diffusion on triangles

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**Exact solutions and ill-posedness**

- FIG. 4 schematic of the Jacquard and Segurie exact result for the growth of a single viscous finger.

- FIG. 5 schematic of numerical concentration profile at zero and low dispersion obtained with initial conditions like that of FIG 3.
\( \mu_0 = 10.0 \)
\( \mu_s = 1.0 \)

\( d_L = 0 \)
\( d_T = 0 \)

Run 2

\( \mu_0 = 10.0 \)
\( \mu_s = 1.0 \)

\( d_L = 0.001 \)
\( d_T = 0.0001 \)

Run 1

\( t = 0 \)
\( t = 20 \)
\( t = 40 \)
\( t = 60 \)
\[ d_L = 0.001 \]
\[ d_T = 0.0001 \]

\( t = 0 \)

\( t = 3 \)
Motion of a viscous filament: A very simple case

\[ p = P(t) \quad p = 0 \]

\[ \mu_s = 0 \]

\[ \mu_o \]

Motion of a viscous filament

\[ \lambda(\xi, t) \]

Conservation of mass

\[ \frac{\partial}{\partial t} \int_{\xi_0}^{\xi_1} \lambda(\xi, t)(x^2_{\xi} + y^2_\xi)^{1/2} \, d\xi = 0 \]

Darcy’s law

\[ \frac{\partial \mathbf{x}}{\partial t} = -\frac{k\Delta P}{\mu_o \lambda} \mathbf{n} \]
Motion of a viscous filament

\[
\frac{\partial}{\partial t} \int_{\xi_0}^{\xi_1} \lambda(\xi, t)(x_\xi^2 + y_\xi^2)^{1/2} \, d\xi = 0
\]

\[
\frac{\partial}{\partial t} [\lambda(\xi, t)(x_\xi^2 + y_\xi^2)^{1/2}] = 0
\]

\[
\lambda(\xi, t)(x_\xi^2 + y_\xi^2)^{1/2} = \alpha(\xi)
\]

Motion of a viscous filament

\[ t = (x_\xi, y_\xi), \quad n = \frac{(y_\xi, -x_\xi)}{(x_\xi^2 + y_\xi^2)^{1/2}}, \quad n.n = 1, \quad n.t = 0 \]

\[
\frac{\partial x}{\partial t} = -\frac{k\Delta P}{\mu_o \lambda} n
\]

\[
= -\frac{k\Delta P}{\mu_o \lambda} \frac{(y_\xi, -x_\xi)}{(x_\xi^2 + y_\xi^2)^{1/2}}
\]

\[
= -\frac{k\Delta P}{\mu_o \alpha} (y_\xi, -x_\xi)
\]

\[
\frac{\partial x}{\partial t} = a(\xi) \frac{\partial y}{\partial \xi}
\]

\[
\frac{\partial y}{\partial t} = -a(\xi) \frac{\partial x}{\partial \xi}
\]
Motion of a viscous filament

\[
\frac{\partial x}{\partial t} = a(\xi) \frac{\partial y}{\partial \xi} \\
\frac{\partial y}{\partial t} = -a(\xi) \frac{\partial x}{\partial \xi}
\]

\[a(\xi) = 1\]

Viscous filaments = Cauchy-Riemann equations

Solution 1.
\[
x = t \\
y = \xi
\]

Solution 2.
\[
x = e^{nt} \cos(n \xi) \\
y = e^{nt} \sin(n \xi)
\]

\[\therefore \text{Small wavelengths grow faster than long wavelengths} \]
\[
(n = \frac{2\pi}{l})
\]
Numerical simulations

\( \mu_0 = 10.0 \)

\( \mu_s = 1.0 \)

Numerical simulations

\( \mu_0 = 10^6 \)

\( \mu_s = 1.0 \)

\( d_L = 0.001 \)

\( d_T = 0.001 \)

\( k_x = 1.0 \)

\( k_y = 1.0 \)
\[ \mu_o = 10^6 \]
\[ \mu_s = 1.0 \]
\[ d_L = 0.001 \]
\[ d_T = 0.001 \]
\[ k_x = 1.0 \]
\[ k_y = 1.0 \]

Finite Volume simulations by R. Booth, 2007
The Koval Model

\[ \bar{c}(x, t) = \frac{1}{W} \int c(x, y, t) dy \]

\[ \varphi \frac{\partial \bar{c}}{\partial t} + \nabla \cdot (\bar{u} f(\bar{c})) = 0 \]

\[ \nabla \cdot \bar{u} = 0, \quad \bar{u} = -\frac{K}{\tilde{\mu}(\bar{c})} \nabla \bar{p} \]

CFD analysis for the Koval model

Figure 3.5: Numerical simulations of solutions of (3.23) in the moving frame \( = x - t \).
Red represents the less viscous solvent \( (c = 1) \) and blue represents the more viscous oil \( (c = 0) \). The dashed, black lines represent the locations of the shocks representing the root and tip regions. We have taken \( M = 2 \) and \( \text{Pe} = 1/100 \) for this simulation.
Concluding Discussion

- Viscous fingering is a fascinating area, and of practical importance.
- Even very simple models can display fingering - the Cauchy-Riemann equations provide a simple model.
- Simulations with low numerical diffusion will mimic reality.
- Simulation helps to determine effective medium models such as the Koval Model.

References


CLF. A Moving Point Method for Arbitrary Peclet Number Multi-Dimensional Convection-Diffusion Equations.


