Optimal Control and Estimation: An ICFD Retrospective

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Outline

1. Introduction
2. Tidal Energy
3. Flight Control
4. Numerical Weather Prediction
5. Conclusions / Future
1. Introduction
Dynamic System

Input \( u \)

System State \( x \)

Output \( y \)

Input \( u \) is known/measurable
Output \( y \) is known/measurable
State \( x \) is not known/not measurable
**Dynamical Models**

- Continuous vs. Discrete
  - ODE
  - PDE

- Deterministic vs. Stochastic
  - Linear
  - Nonlinear

- Automatic Feedback vs. Optimal Feedback
Control problem:
Find the input or control function that produces a given response from the system

State estimation problem:
Given measured output from system, find the state of the system at a specified time.
2. Tidal Energy
Simple Tidal Power Model

CONTROL PROBLEM

\[
\max_{\alpha} E(\alpha) = \int_0^1 \alpha(t) \left[ f(t) - \eta(t) \right]^2 \, dt
\]

subject to

\[
\dot{\eta} = K \alpha(t) \left[ f(t) - \eta(t) \right], \quad K = hT/l
\]

\[
\eta(0) = \eta(1)
\]

and

\[
0 \leq \alpha \leq a_0, \quad \nu \in [0,1]
\]

Take

\[
f(t) = A \cos \omega t, \quad \omega = \frac{2\pi}{T}
\]
Necessary Conditions - Tidal Problem

\[ L(\beta) - L(\alpha) = \int_0^1 \left( [f - \eta]^2 + \kappa \lambda [f - \eta] \right) (\beta - \alpha) \, dt \]

\[ + o(1|\beta - \alpha|) \leq 0 \]

where

\[ \eta = K_2 [f - \eta] \quad \text{(state)} \]

\[ \lambda = K_2 \lambda + 2 \alpha [f - \eta] \quad \text{(Adjoint)} \]

\[ \eta(0) = \eta(1), \quad \lambda(0) = \lambda(1) \quad \text{(B.C.)} \]

\[ \Rightarrow \]

\[ a = a_0 \quad \text{if} \quad \forall E \geq [f - \eta]^2 + \kappa \lambda [f - \eta] > 0 \]

\[ = 0 \quad \text{if} \quad \ll 0. \]

Control is discontinuous!
Results

No control

Energy = 0.1212

With control

Switchpoints as follows:

<table>
<thead>
<tr>
<th>J</th>
<th>T(j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1024</td>
</tr>
<tr>
<td>2</td>
<td>0.3974</td>
</tr>
<tr>
<td>3</td>
<td>0.6026</td>
</tr>
<tr>
<td>4</td>
<td>0.8974</td>
</tr>
</tbody>
</table>

Energy = 0.2276
Severn Barrage Model

Plan of model estuary

Section of estuary

Cross section of estuary

Fig. 1

Geometry of model estuary
Full Non-Linear Dynamic (PDE) Model

Shallow Water Equations

\[ b(\eta, x) \eta_t = -\frac{d}{dx} \left( A(\eta, x) u \right) \]
\[ u_t = -g \eta_x - \frac{g n^2 u u_x}{r_m(\eta, x)} \]
\[ \eta(-L_1, t) = f(t), \quad A(\eta, L_2) u(L_2, t) = 0 \]
\[ A(\eta^+, 0^+) u(0^+, t) = A(\eta^-, 0^-) u(0^-, t) = Q(t) \]

where \( Q(t) = a_1 k_1 P(t) + a_2 k_2 R(t) \)
\[ H = \eta(0^-, t) - \eta(0^+, t) \]
\[ \eta(x, 0) = \eta(x, T) \quad u(x, 0) = u(x, T) \]

Average Revenue

\[ E = \frac{1}{T} \int_0^T C(t) \alpha(t) k_1 F(t) \, dt \]

to be maximized

subj to \( 0 \leq \alpha_1, \alpha_2 \leq 1 \)
## Average Power Output (GW)

### Ebb Scheme

**Spring (8.5)**
- **NODE**: 2.52
- **LPDE**: 2.43
- **NPDE**: 2.44

**Neap (4.5)**
- **NODE**: 0.67
- **LPDE**: 0.60
- **NPDE**: 0.64

### Two Way Scheme

**Spring (8.5)**
- **NODE**: 2.52
- **LPDE**: 2.46
- **NPDE**: 2.57

**Neap (4.5)**
- **NODE**: 0.67
- **LPDE**: 0.68
- **NPDE**: 0.80
<table>
<thead>
<tr>
<th></th>
<th>Weighted Output (NPDE)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Av. Power (GW)</strong></td>
<td></td>
</tr>
<tr>
<td>$C(t) = 1$</td>
<td>1.42</td>
</tr>
<tr>
<td>$C(t) = \text{Winter Tariff}$</td>
<td>1.38</td>
</tr>
<tr>
<td><strong>Total Revenue (Units)</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.05</td>
</tr>
<tr>
<td></td>
<td>(£17.0m)</td>
</tr>
<tr>
<td></td>
<td>5.25</td>
</tr>
<tr>
<td></td>
<td>(£17.6m)</td>
</tr>
</tbody>
</table>
3.
Automatic Flight Control Systems
Model Equations

• Linearize about "trim conditions"
  = steady-state, symmetric level flight

• Implies Time-Invariant linear equations for perturbations from trim

• Separates systems for
  - Longitudinal Motion
    \((u, w, q, \theta)\)
  - Lateral Motion
    \((v, p, r, q, \psi)\)
Dynamic System

\[ \dot{x} = Ax + By \]
\[ y = Cx \]
Dynamic System

\[
\text{Input } y \\
\text{System State } x \\
\text{Output } y \\
\]

\[
y = Ky + r
\]

Feedback System:

\[
\dot{x} = Ax + BK_y + Br \\
= (A + BK_C)x + Br \\
y = Cx
\]
Objective

Choose $K$ to:

- Stabilize/place poles
- Decouple input/output - assign eigenvectors
- Ensure robustness (insensitivity)
THEOREM

Given $A, B, C$, $\Lambda = \text{diag}\{\lambda_i\}$
and $V = [v_1, v_2, \ldots, v_n]$ non-singular

Then there exists $K$ s.t.

$$(A + BKC)V = V\Lambda$$

if and only if

$$v_i = v_{0i} e_i$$
$$e_i \in \mathbb{R} \{ (I - BB^*) (A - \lambda_i I) \}$$

$$x^T = e_i^T v^T e J = \mathbb{N}_c \{ (A - \lambda_i I) x I - C^* C \}.$$
Measures

**Input Coupling**

$$J_i = \|G^T - W_i^T B\|_F^2$$

**Robustness**

$$J_2 = \|V^{-1}\|_F^2 \quad \text{where} \quad \|V_i\|_F = 1$$

**Pole Accuracy** = distance of $W_i^T$ from $J_i$

$$J_3 = \sum_i \beta_i \|W_i^T \tilde{T}_i\|_2^2$$

where $\tilde{T}_i$ is basis for $J_i$.

Assume:

$$W^T = \begin{bmatrix} W_1^T \\ W_2^T \end{bmatrix} = [V_1, V_2]^T = V^{-1}$$
Aim:

Given $V_i$ and $\omega_j$, $j = 1, 2, 3$:

$$\min J = \left\{ \omega_1^2 J_1 + \omega_2^2 J_2 + \omega_3^2 J_3 \right\}$$

subject to

$V_i \in \Delta_i$

$\|V_i\|_1 = 1$

$i = p+1, p+2, \ldots, n$

Theorem

$\min J$ for each $i$ is equivalent to solving a standard least squares problem.
Bifurcation Diagrams - Jet Fighter

No control

Robust decoupling control

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4. Numerical Weather Prediction
Significant Properties:

- Very large number of unknowns ($10^7 - 10^8$)
- Few observations ($10^5 - 10^6$)
- System nonlinear unstable/chaotic
- Multi-scale dynamics
- Stochastic errors in model and observations
State Estimation

Aim:

Find the best estimate (analysis) of the true state of a system, consistent with both observations and the system dynamics given:

- Numerical prediction model
- Observations of the system (over time)
- Background state (prior)
- Estimates of the errors
State Estimation

Aim:

Find the best estimate (analysis) of the true state of a system, consistent with both observations and the system dynamics:

- Numerical prediction model
- Observations of the system (over time)
- Background state (prior)
- Estimates of the errors
Best Unbiased Estimate

\[
\begin{align*}
\min \quad & J(x_0) = \frac{1}{2}(x_0 - x_b)^T B^{-1}(x_0 - x_b) \\
+ & \sum_{i=0}^{n} (H_i[x_i] - y_i^o)^T R_i^{-1}(H_i[x_i] - y_i^o)
\end{align*}
\]

subject to \( x_{i+1} = M_i(x_i) \)

where

- \( x_b \) - Background state (prior estimate)
- \( y_i \) - Observations
- \( H_i \) - Observation operator
- \( B \) - Background error covariance matrix
- \( R_i \) - Observation error covariance matrix
Algorithm

• Uses Approximate Gauss-Newton method:
  – Solves sequence of linearized least squares problems by inner gradient iteration procedure
  – Finds gradients by adjoint integration
  – Truncates inner loop iterations
  – Uses approximate linear system models

• Theoretical convergence results obtained by reference to Gauss-Newton method (SIOPT).
New Research

• Find approximate linear system models using optimal reduced order modeling techniques from control theory to improve computational efficiency

• Test feasibility of approach: compare solutions using
  – Low resolution linear model
  – Optimal reduced order model

for a simple shallow water flow model.
Norm of analysis error for 1-D SWE model

Low Res Model of order = 200
Norm = 0.2112

vs Reduced Model of order = 200
Norm = 0.0027

Low Res Model of order = 200
Norm = 0.2112

vs Reduced Model of order = 80
Norm = 0.1726

Red (dotted) = Low Resolution Model
Green (dashed) = Reduced Order Model

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5. Conclusions / Future

• Many exciting open problems exist with valuable applications

• New research areas include:
  - PDE constrained optimization
  - Stochastic PDE control and estimation (data assimilation)
  - Model Order Reduction
  - Uncertainty in prediction
References


