DG methods for aerodynamic flow simulations:
Higher order, error estimation and adaptivity

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- structured grid generator: MegaCads
- structured Finite Volume flow solver: Flower
- unstructured Finite Volume flow solver: TAU

used in industry: Airbus, EADS-M, ...
Motivation

Higher order methods:

- Numerical resolution and tracking of vortices
  - Helicopters: Vortex creation and blade-vortex interaction
  - Transport aircrafts: wake-vortices
- Numerical resolution of viscous boundary layers
- Numerical approximation of aerodynamical forces: lift, drag, moments

Error estimation:
- Reliable prediction of aerodynamical forces

Adaptivity:
- Mesh refinement for better resolution of vortices, boundary layers, etc.
- Goal-oriented mesh refinement for accurate approximation of aerodynamical forces
Overview

▶ The Discontinuous Galerkin discretization of compr. Euler & Navier-Stokes
▶ Higher order computational results
▶ A posteriori error estimation
▶ Isotropic and anisotropic goal-oriented (adjoint-based) adaptivity
The Discontinuous Galerkin discretization
of the compressible Euler & Navier-Stokes equations
The compr. Euler and Navier-Stokes equations in 2D

The compressible Euler equations:

\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho E \end{pmatrix} + \frac{\partial}{\partial x_1} \begin{pmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ v_1(\rho E + p) \end{pmatrix} + \frac{\partial}{\partial x_2} \begin{pmatrix} \rho v_2 \\ \rho v_2^2 + p \\ \rho v_1 v_2 \\ v_2(\rho E + p) \end{pmatrix} = 0
\]
The compr. Euler and Navier-Stokes equations in 2D

The compressible Euler equations:

\[
\frac{\partial}{\partial t} \begin{pmatrix}
\rho \\
\rho v_1 \\
\rho v_2 \\
\rho E
\end{pmatrix} + \frac{\partial}{\partial x_1} \begin{pmatrix}
\rho v_1 \\
\rho v_1^2 + p \\
\rho v_1 v_2 \\
v_1(\rho E + p)
\end{pmatrix} + \frac{\partial}{\partial x_2} \begin{pmatrix}
\rho v_2 \\
\rho v_1 v_2 \\
\rho v_2^2 + p \\
v_2(\rho E + p)
\end{pmatrix} = 0
\]

\[
\frac{\partial u}{\partial t} + \nabla \cdot \mathcal{F}^c(u) = 0
\]
The compr. Euler and Navier-Stokes equations in 2D

The compressible Euler equations:

\[
\frac{\partial}{\partial t} \begin{pmatrix}
\rho \\
\rho v_1 \\
\rho v_2 \\
\rho E
\end{pmatrix} + \frac{\partial}{\partial x_1} \begin{pmatrix}
\rho v_1 \\
\rho v_1^2 + p \\
\rho v_1 v_2 \\
v_1 (\rho E + p)
\end{pmatrix} + \frac{\partial}{\partial x_2} \begin{pmatrix}
\rho v_2 \\
\rho v_1 v_2 \\
\rho v_2^2 + p \\
v_2 (\rho E + p)
\end{pmatrix} = 0
\]

\[
\frac{\partial}{\partial t} \mathbf{u} + \nabla \cdot \mathbf{F}^c(\mathbf{u}) = 0
\]

The compressible Navier-Stokes equations:

\[
\frac{\partial}{\partial t} \mathbf{u} + \nabla \cdot \mathbf{F}^c(\mathbf{u}) - \nabla \cdot \mathbf{F}^v(\mathbf{u}, \nabla \mathbf{u}) = 0
\]
The compr. Euler and Navier-Stokes equations in 2D

The compressible Euler equations:

\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho E \end{pmatrix} + \frac{\partial}{\partial x_1} \begin{pmatrix} \rho v_1 \\ \rho v_1 v_2 + p \\ \rho v_1 v_2 \\ v_1(\rho E + p) \end{pmatrix} + \frac{\partial}{\partial x_2} \begin{pmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ v_2(\rho E + p) \end{pmatrix} = 0
\]

\[
\frac{\partial}{\partial t} u + \nabla \cdot \mathcal{F}^c(u) = 0
\]

The compressible Navier-Stokes equations:

\[
\frac{\partial}{\partial t} u + \nabla \cdot \mathcal{F}^c(u) - \nabla \cdot \mathcal{F}^v(u, \nabla u) = 0
\]

\[
f_1^v(u, \nabla u) = \begin{pmatrix} 0 \\ \tau_{11} \\ \tau_{21} \\ \tau_{11} v_1 + \tau_{12} v_2 + \kappa T x_1 \end{pmatrix}, \quad f_2^v(u, \nabla u) = \begin{pmatrix} 0 \\ \tau_{12} \\ \tau_{22} \\ \tau_{21} v_1 + \tau_{22} v_2 + \kappa T x_2 \end{pmatrix}.
\]
DG discretization of the compr. Euler equations

The problem:

\( \nabla \cdot F^c(u) = 0 \quad \text{in} \ \Omega, \)

with \( u = (\varrho, \varrho v_1, \varrho v_2, \rho E)^T. \)

The discretization of DG\((p)\): Find \( u_h \) in \( V_{h,p} \) such that

\[
\mathcal{N}(u_h, v_h) \equiv \sum_{\kappa \in T_h} \left\{ - \int_{\kappa} F^c(u_h) : \nabla v_h \, dx + \int_{\partial \kappa \setminus \Gamma} \mathcal{H}(u_h^+, u_h^-, n_\kappa) \, v_h^+ \, ds \right\}
+ \int_{\Gamma} \mathcal{H}(u_h^+, u_{\Gamma}(u_h^+), n_\kappa) \, v_h^+ \, ds = 0 \quad \forall v_h \in V_{h,p},
\]

with

\[
V_{h,p} = \left\{ v \in [L_2(\Omega)]^4 : v|_{\kappa} \in [Q_p(\kappa)]^4 \quad \forall \kappa \in T_h \right\},
\]
DG discretization of the compr. Euler equations

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with \( u = (\varrho, \varrho v_1, \varrho v_2, \rho E)^T \).

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+ \int_{\Gamma} \mathcal{H}(u_h^+, u_h^+(\kappa), n_{\kappa}) v^+_h \, ds = 0 \quad \forall v_h \in V_{h,p},
\]

with

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\]
DG discretization of the compr. Euler equations

The problem:

$$\nabla \cdot \mathcal{F}^c(u) = 0 \quad \text{in } \Omega,$$

with \(u = (\rho, \rho v_1, \rho v_2, \rho E)^T\).

The discretization of DG(\(p\)): Find \(u_h\) in \(V_{h,p}\) such that

\[
\mathcal{N}(u_h, v_h) \equiv \sum_{\kappa \in T_h} \left\{ - \int_{\kappa} \mathcal{F}^c(u_h) : \nabla v_h \, dx + \int_{\partial \kappa \setminus \Gamma} \mathcal{H}(u_{h}^+, u_{-}^+, n_{\kappa}) \, v_{h}^+ \, ds \right\} \\
+ \int_{\Gamma} \mathcal{H}(u_{h}^+, u_{\Gamma}(u_{h}^+), n_{\kappa}) \, v_{h}^+ \, ds = 0 \quad \forall v_h \in V_{h,p},
\]

with

\[
V_{h,p} = \left\{ v \in [L^2(\Omega)]^4 : v|_{\kappa} \in [Q_p(\kappa)]^4 \quad \forall \kappa \in T_h \right\},
\]
DG discretization of the compr. Navier-Stokes equations

Find $u_h$ in $V_{h,p}$ such that

$$N(u_h, v_h) \equiv - \int_{\Omega} \mathcal{F}^c(u_h) : \nabla_h v_h \, dx + \sum_{\kappa \in T_h} \int_{\partial_k \Gamma} \mathcal{H}(u^+_h, u^-_h, n_k) \cdot v^+_h \, ds$$
$$+ \int_{\Omega} \mathcal{F}^v(u_h, \nabla_h u_h) : \nabla_h v_h \, dx - \int_{\Gamma_I} \{ \{ G(u_h) \nabla u_h \} \} : [v_h] \, ds$$
$$- \int_{\Gamma_I} \{ \{ G^\top(u_h) \nabla v_h \} \} : [u_h] \, ds + \int_{\Gamma_I} \delta(u_h) : [v_h] \, ds + N_{\Gamma}(u_h, v_h) = 0$$

for all $v_h$ in $V_{h,p}$, where the penalization term is given by

$$\delta(u_h) = \tilde{C}_{ip} p_h^2 [u_h] \quad \text{for the standard IP scheme, used in [Hartmann,Houston2006],}$$
$$\delta(u_h) = C_{ip} p_h^2 \{ \{ G(u_h) \} \} [u_h] \quad \text{for the new IP scheme, see [Hartmann,Houston2007],}$$
$$\delta(u_h) = \eta_e \{ L_0^e(u_h) \} \quad \text{for the BR2 scheme, see [Bassi,Rebay et al. 1997],}$$

and the local lifting operator is defined by: find $L_0^e(u_h) \in \Sigma_{h,p}$ such that

$$\int_{\Omega} L_0^e(u_h) : \tau \, dx = \int_{e} [u_h] : \{ \{ G^\top(u_h) \tau \} \} \, ds \quad \forall \tau \in \Sigma_{h,p}.$$
DG discretization of the compr. Navier-Stokes equations

Find $u_h$ in $V_{h,p}$ such that, with

$$
\mathcal{N}_{\Gamma}(u_h, v_h) \equiv \int_{\Gamma} \mathcal{H}_{\Gamma}(u^+_h, u_{\Gamma}(u^+_h), n) \cdot v^+_h \, ds + \int_{\Gamma} \delta_{\Gamma}(u^+_h) \cdot v^+_h \, ds,
$$

$$
- \int_{\Gamma} G_{\Gamma}(u_h) \nabla u_h : [v_h] \, ds
$$

$$
- \int_{\Gamma} G^T_{\Gamma}(u_h) \nabla v_h : (u^+_h - u_{\Gamma}(u^+_h)) \otimes n \, ds,
$$

where the boundary function $u_{\Gamma}(u_h)$ realizes the following boundary conditions:

- supersonic in- or outflow
- subsonic in- or outflow
- noslip wall boundary condition (adiabatic or isothermal)

Symmetry boundary conditions according to the discretization on interior faces.
Optimal order DG discretizations: adjoint consistency

For adjoint consistency require, see [Lu,Darmofal2006],

\[ \mathcal{H}_\Gamma(u^+_h, u^+_\Gamma, n) = n \cdot \mathcal{F}^c_\Gamma(u^+_h) = n \cdot \mathcal{F}^c(u^+_\Gamma), \quad \text{on } \Gamma_W. \]

and the aerodynamical force coefficients must be evaluated as, see [Hartmann2007],

\[ \tilde{J}(u_h) = J(u^\Gamma(u_h)) + \int_\Gamma \delta_\Gamma(u_h) : z_\Gamma \otimes n \, ds, \]

where \( J(u) \) represents the lift or drag coefficient:

\[ J(u) = \int_{\Gamma_W} (p \, n - \tau \, n) \cdot \psi_{\Gamma_W} \, ds. \]

Modification of a specific \( J(u) \) was originally given for Poisson’s equation in [Harriman,Gavaghan,Süli2004].
Higher order computational results
Laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ flow around the NACA0012 airfoil
Laminar test case

\( M = 0.5, \ Re = 5000, \ \alpha = 0 \) flow around the NACA0012 airfoil

Grid of 3072 cells:

Zoom of this grid:
Laminar test case

\[ M = 0.5, \, Re = 5000, \, \alpha = 0 \] flow around the NACA0012 airfoil

Grid of 3072 cells:

Zoom of this grid:

Computation using DG(\(p\)), \(p = 1, 2, 3\) on sequence of globally refined grids of 3072, 12288, 49152 and 196608 cells.
Higher order computations for laminar test case

\[ M = 0.5, \ Re = 5000, \ \alpha = 0 \] flow around the NACA0012 airfoil

Convergence of cdp and cdf under global refinement, see [Hartmann,Houston2006]

cdp (pressure induced drag)
Higher order computations for laminar test case

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Convergence of cdp and cdf under global refinement, see [Hartmann, Houston2006]
Convergence of cdp

cdp

cdp

cells

DG(3), global refinement
DG(2), global refinement
DG(1), global refinement
reference cdp
Convergence of cdp

**cdp**

![Graph of cdp vs. number of cells for different DG orders and global refinement]

**reference cdp - cdp**

![Graph of reference cdp vs. number of cells for DG(1) global refinement]

- DG(3), global refinement
- DG(2), global refinement
- DG(1), global refinement
- Reference cdp
Convergence of cdp

**cdp**

**reference cdp - cdp**
Convergence of cdp

cdp

reference cdp - cdp
Convergence of cdp

**cdp**

- DG(3), global refinement
- DG(2), global refinement
- DG(1), global refinement

**reference cdp - cdp**

- DG(3), global refinement
- DG(2), global refinement
- DG(1), global refinement

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<tr>
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Deutsches Zentrum für Luft- und Raumfahrt e.V.
in der Helmholtz-Gemeinschaft
Convergence of cdp

cdp

reference cdp - cdp
Convergence of cdp

cdp

0.024
0.022
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006

10000
100000

cells

DG(3), global refinement
DG(2), global refinement
DG(1), global refinement
reference cdp

0.006
0.008
0.01
0.012
0.014
0.016
0.018
0.02
0.022
0.024

10000
100000

cells

reference cdp - cdp

0.001
0.01

10000
100000

cells

DG(1), global refinement
DG(2), global refinement
DG(3), global refinement

3.072 cells

196.608 cells
Convergence of cdp

cdp

Reference cdp - cdp

3.072 cells
196.608 dof

196.608 cells
3.145.728 dof
Convergence of cdp

**cdp**

- **DG(3), global refinement**
- **DG(2), global refinement**
- **DG(1), global refinement**
- **reference cdp**

**reference cdp - cdp**

- **DG(1), global refinement**
- **DG(2), global refinement**
- **DG(3), global refinement**

3.072 cells
196.608 dof
8 min

196.608 cells
3.145.728 dof
136 min
Convergence of cdf

cdf

![Graph showing convergence of cdf for reference cdf and DG(1), DG(2), DG(3) with global refinement.](image)
Convergence of cdf

cdf

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DG(1), global refinement
DG(2), global refinement
DG(3), global refinement
Convergence of cdf

cdf

cdf - reference cdf
Convergence of cdf

cdf

cdf - reference cdf
Convergence of cdf

cdf

cdf - reference cdf

3.072 cells

196.608 cells
Convergence of cdf

**cdf**

![CDF plot](image1)

**cdf - reference cdf**

![CDF difference plot](image2)

3.072 cells
196.608 dof

196.608 cells
3.145.728 dof
Convergence of cdf

cdf

cdf - reference cdf

3.072 cells 196.608 dof 8 min
196.608 cells 3.145.728 dof 136 min
Convergence of cdf

CDF

CDF - reference CDF

3.072 cells
196,608 dof
8 min

196,608 cells
3,145,728 dof
136 min

786,432 cells
12,582,912 dofs
\approx 12 \text{ h (extrapolated)}
Higher order approximation of viscous boundary layers
Higher order approximation of viscous boundary layers

Flat plate problem: $M = 0.01$, $Re = 10000$, see [Hartmann,Houston2006]
Higher order approximation of viscous boundary layers

Flat plate problem: $M = 0.01$, $Re = 10000$

Approximation of viscous force exerted on wall up to 5% requires

<table>
<thead>
<tr>
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<th>DG(1)</th>
<th>DG(2)</th>
<th>DG(3)</th>
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<tr>
<td>elements</td>
<td>36</td>
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<tr>
<td>DoF</td>
<td>72</td>
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orthogonal to the wall
Laminar Delta Wing

ADIGMA
BTC-2
Test case

\[ M = 0.3, \]
\[ \alpha = 12.5^\circ, \]
\[ Re = 4000, \]

isothermal
noslip wall boundary
Laminar Delta Wing

\[ M = 0.3, \alpha = 12.5^\circ, \]
\[ Re = 4000 \]

3264 elements on the half domain

left: DG(1), 2nd order
right: DG(4), 5th order
Laminar Delta Wing

\[ M = 0.3, \alpha = 12.5^\circ, \]
\[ Re = 4000 \]

3264 elements on the half domain

left: DG(1), 2nd order
right: DG(4), 5th order

DG(1), 40 DoFs/cell:
130,560 DoFs

DG(4), 625 dofs/cell:
2,040,000 DoFs
Laminar Delta Wing, $M = 0.3, \alpha = 12.5^\circ, Re = 4000$

Convergence of $c_l$ on a sequence of globally refined (non-nested) meshes.
Laminar Delta Wing, $M = 0.3, \alpha = 12.5^\circ, Re = 4000$

Convergence of $c_l$ on a sequence of globally refined (non-nested) meshes.
Laminar Delta Wing

DG(3) discretization on locally refined grid of 8,122 elements (half domain)
Delta Wing, laminar flow, $M = 0.3, \alpha = 12.5^\circ, Re = 4000$

Cut along one of the vortices
Delta Wing, laminar flow, $M = 0.3, \alpha = 12.5^\circ, Re = 4000$

DG discretization on locally refined grid of 8.122 elements (half domain)

- $p=1$
- $p=2$
- $p=3$
Error estimation
Error estimation for single target quantities

Given a discretization: find $u_h \in V_h$ such that

$$\mathcal{N}(u_h, v_h) = 0 \quad \forall v_h \in V_h.$$  \hspace{1cm} (1)

and a target quantity $J$.

Computed: $J(u_h)$, exact (but unknown): $J(u)$, what is $J(u) - J(u_h)$?!

By employing a duality argument obtain (approximate) error representation

$$J(u) - J(u_h) = -\mathcal{N}(u_h, z) = \mathcal{R}(u_h, z)$$

$$\approx \mathcal{R}(u_h, \tilde{z}_h) = \sum_{\kappa \in T_h} \eta_{\kappa} =: \eta,$$

Discrete adjoint problem: find $\tilde{z}_h \in \tilde{V}_h$ such that

$$\mathcal{N}'[u_h](w_h, \tilde{z}_h) = J'[u_h](w_h) \quad \forall w_h \in \tilde{V}_h.$$
Error estimation for single target quantities: Example

ADIGMA MTC-3, laminar flow, 
$M = 0.5$, $\alpha = 2^\circ$,  
$Re = 5000$

Mach number isolines

<table>
<thead>
<tr>
<th>Force coefficients</th>
<th>required accuracy</th>
</tr>
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<tbody>
<tr>
<td>pressure induced drag coefficient: $J(u) = c_{dp}$</td>
<td>5e-4</td>
</tr>
<tr>
<td>viscous drag coefficient: $J(u) = c_{df}$</td>
<td>5e-4</td>
</tr>
<tr>
<td>total lift coefficient: $J(u) = c_l$</td>
<td>5e-3</td>
</tr>
<tr>
<td>total moment coefficient: $J(u) = c_m$</td>
<td>5e-4</td>
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</table>
Error estimation for single target quantity: $J(u) = c_{dp}$

Example: MTC-3, laminar flow, $M = 0.5$, $\alpha = 2^\circ$, $Re = 5000$

Target quantity: $J(u) = c_{dp}$ (pressure induced drag), Ref. value: $J_{cdp}^{ref}(u) = 0.02380$

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Error estimation for single target quantity: $J(u) = c_{df}$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(u) = c_{df}$ (viscous drag), Ref.value: $J_{cdf}^{ref}(u) = 0.0322835$

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Error estimation for single target quantity: \( J(u) = c_l \)

Example: MTC-3, laminar flow, \( M = 0.5, \alpha = 2^\circ, Re = 5000 \)

Target quantity: \( J(u) = c_l \) (total lift), Ref. value: \( J_{cl}^{ref}(u) = 0.037286 \)

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Error estimation for single target quantity: $J(u) = c_m$

Example: MTC-3, laminar flow, $M = 0.5$, $\alpha = 2^\circ$, $Re = 5000$

Target quantity: $J(u) = c_m$ (total moment), Ref.value: $J_{cm}^{ref}(u) = -0.01661$

<table>
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Error estimation for single target quantities

$z_1$ components of adjoint solutions.  

Top: cdp, cdf;  

Bottom: cl, cm.
Error estimation for multiple target quantities

Given $N$ target quantities $J_i(u), i = 1, \ldots, N$. The **direct approach** requires $N$ discrete adjoint problems: find $\tilde{z}_{i,h} \in \tilde{V}_h$ such that

$$N''[u_h](w_h, \tilde{z}_{i,h}) = J'_i[u_h](w_h) \quad \forall w_h \in \tilde{V}_h, \quad i = 1, \ldots, N,$$

to obtain error estimates for $N$ target quantities

$$J_i(u) - J_i(u_h) = \mathcal{R}(u_h, z_i) \approx \mathcal{R}(u_h, \tilde{z}_{i,h}), \quad i = 1, \ldots, N,$$

The **new approach**, originally developed in [Hartmann,Houston2003] requires one discrete adjoint-adjoint problem (error equation): find $\tilde{e}_h \in \tilde{V}_h$ such that

$$N''[u_h](\tilde{e}_h, w_h) = \mathcal{R}(u_h, w_h) \quad \forall w_h \in \tilde{V}_h,$$

to obtain error estimates for $N$ target quantities

$$J_i(u) - J_i(u_h) \approx J'_i[u_h](e) \approx J'_i[u_h](\tilde{e}_h), \quad i = 1, \ldots, N,$$
Error estimation for multiple target quantities

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

On each mesh compute primal solution $u_h$ and adjoint-adjoint solution $\tilde{e}_h$.

Evaluate exact error: $J^\text{ref}_i(u) - J_i(u_h)$, $i = 1, \ldots, N$,

Evaluate error estimate: $J_i'[u_h](\tilde{e}_h)$, $i = 1, \ldots, N$,

<table>
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Goal-oriented (adjoint-based) refinement
Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil

Mach isolines: close view

Mach isolines: distant view
Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil

Mach isolines: close view

Mach isolines: distant view
Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil

Mach isolines: close view  Mach isolines: distant view

For the efficient and accurate approximation of $J(u) = c_{dp}$:
Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil

For the efficient and accurate approximation of $J(u) = c_d$:

How should the mesh look like?
Residual-based refinement: 
17670 cells with 282720 dofs 
error in $c_{dp}$ : 1.9 $\cdot 10^{-3}$  
error in $c_{df}$ : 1.1 $\cdot 10^{-2}$
Residual-based refinement:
17670 cells with 282720 dofs
error in $c_{dp}$: $1.9 \cdot 10^{-3}$
error in $c_{df}$: $1.1 \cdot 10^{-2}$

Goal-oriented refinement:
10038 cells with 160608 dofs
error in $c_{dp}$: $1.6 \cdot 10^{-4}$
error in $c_{df}$: $7.2 \cdot 10^{-4}$
First summary

Higher order discontinuous Galerkin discretization of the compressible Euler and Navier-Stokes equations

► Same accuracy on coarser meshes & less computational time than for 2nd order
► Accurate error estimation with respect to target quantities
► Efficient adjoint-based (goal-oriented) adaptive mesh refinement
Anisotropic refinement

Use a residual-based or adjoint-based indicator to select the elements to be refined

Use an anisotropic indicator to determine the anisotropic refinement case

Anisotropic indicators:

- **Jump indicator**: jump across a face is connected to approximation quality orthogonal to the face

- **Derivative indicator**: Hessian for 2nd scheme, higher order derivatives for higher order schemes
Anisotropic refinement: Laminar test case

$M = 0.5, Re = 5000, \alpha = 0$ flow around the NACA0012 airfoil

DG(2), i.e. 3rd order, with adjoint-based refinement, error measured in $c_{dp}$

Comparison of:

- isotropic refinement
- anisotropic jump indicator
- anisotropic derivative indicator (3rd derivatives)
Anisotropic refinement: First summary

- Resolution of anisotropic flow features like shocks and boundary layers
- In combination with error estimation and goal-oriented (adjoint-based) refinement: Automatic generation of “optimal” initial layer spacing
Anisotropic refinement: First summary

- Resolution of anisotropic flow features like shocks and boundary layers
- In combination with error estimation and goal-oriented (adjoint-based) refinement: Automatic generation of “optimal” initial layer spacing

Furthermore:
Use anisotropic refinement to optimize grids with inappropriate/bad aspect ratios.
Anisotropic refinement applied to ADIGMA Test Case 1

compressible Euler equations: $M = 0.5, \alpha = 2$:

No anisotropic features
Test case 1: Euler, $M=0.5$, $\alpha = 2^\circ$

Grids: MTC_v1.3_NACA0012_INVISCID (by DLR)

level 6: 112 elements  
level 5: 448 elements  
level 4: 1792 elements  
level 3: 7168 elements  
level 2: 28672 elements  
level 1: 114688 elements
Test case 1: Euler, $M=0.5, \alpha = 2^\circ$

Coarse grid:

Bad aspect ratios will be inherited to isotropically refined elements
Test case 1: Euler, M=0.5, $\alpha = 2^\circ$

Goal-oriented (adjoint-based) refinement: target quantity $J(u) = c_{lp}$

isotropic (16282 elements)
error in $c_{lp}$: $1.4 \times 10^{-4}$

anisotropic (5403 elements)
error in $c_{lp}$: $1.1 \times 10^{-4}$
Outlook: Next steps towards application in industry

- Extension of the flow solver
  - to three-dimensional turbulent, high Reynolds flow
- The above extension also for
  - the computation of the adjoint solution
  - the evaluation of refinement indicators (residual-based and adjoint-based)
  - and the error estimation with respect to aerodynamical force coefficients
- Error estimation and adaptivity with respect to multiple target quantities
- Extension to hybrid meshes
- Extension to hp-refinement
- Efficient solution algorithms
  - linear and nonlinear multigrid
  - h- and p- multigrid
EU Project: ADIGMA

Adaptive Higher-Order Variational Methods for Aerodynamic Applications in Industry

Start was 1st of Sept 2006

Co-ordinator: DLR
Industrial partners: Airbus-D, Airbus-F, Dassault, Alenia, EADS-M
Research institutes: DLR, ONERA, NLR, FOI, INRIA, VKI
Universities: Uni Bergamo, Uni Twente, Uni Swansea, Uni Nottingham, Uni Stuttgart, Uni Warsaw, Uni Prague, ENSAM, (Uni Nanjing)
SMEs: ARA, CENAERO

Topics: Mainly Discontinuous Galerkin methods, also Residual Distribution Schemes, Multigrid, Newton-like methods, adaptation, error estimation, hp-refinement